COMPLEXITY IN A COURNOT DUOPOLY GAME WITH DIFFERENTIATED GOODS BETWEEN SEMI-PUBLIC AND PRIVATE FIRMS

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Abstract: *In this paper we analyze the dynamics of a nonlinear Cournot-type duopoly game with differentiated goods for two bounded rational players with different objective functions. Specifically, the first player is a semi-public company and cares about a percentage of the social welfare and the second player is a private company which cares only about its own profit maximization. The game is modelled with a system of two difference equations We examine the effect of the parameters on the equilibria of the model and we analyse their stability conditions. Complex dynamic features including period doubling bifurcations of the unique Nash equilibrium are also investigated. Numerical simulations are carried out to show the complex behaviour. The chaotic features are justified numerically via computing Lyapunov numbers, sensitive dependence on initial conditions, bifurcation diagrams and strange attractors.*

Key words*:* Cournot duopoly game; Social welfare; Discrete dynamical system; Nash equilibrium; Stability; Bifurcation diagrams; Lyapunov numbers; Strange attractors; Chaotic Behavior.

JEL Classification Codes : C62, C72, D43.

1. INTRODUCTION

Duopoly game is the most basic form of oligopoly, a market dominated by a small number of companies. Cournot, in 1838 has introduced the first formal theory of oligopoly. In 1883 another French mathematician Joseph Louis Francois Bertrand modified Cournot game suggesting that firms actually choose prices rather than quantities. Originally Cournot and Bertrand models were based on the premise that all players follow naïve expectations, so that in every step, each player (firm) assumes the last values that were taken by the competitors without estimation of their future reactions. However, in real market conditions such an assumption is very unlikely since not all players share naive beliefs. Therefore, different approaches to firm behavior were proposed. Some authors considered duopolies with homogeneous expectations and found a variety of complex dynamics in their games, such as appearance of strange attractors (Agiza, 1999, Agiza et al., 2002, Agliari et al., 2005, 2006, Bischi, Kopel, 2001, Kopel, 1996, Puu, 1998, Sarafopoulos, 2015b, Sarafopoulos *et al.,* 2019a). Also models with heterogeneous agents were studied (Agiza, Elsadany , 2003, 2004, Agiza et al., 2002, Den Haan , 20013, Fanti, Gori, 2012, Sarafopoulos, 2015a, Sarafopoulos *et al.,*2017, 2018, 2019b Tramontana , 2010, Zhang , 2007).When bounded rational and adaptive expectations are chosen, the nonlinear models become complicated and no analytical tool are available. This issue has been previously analyzed by Baumol and Quandt, 1964, Puu 1995, Naimzada and Ricchiuti, 2008, Askar, 2013, Askar, 2014, Agiza, Elsadany, 2004, Naimzada, Sbragia, 2006, Zhang *et al*, 2007, Askar, 2014.

All related literature analyzes firm's dynamic behavior by assuming a private oligopoly where they are merely keen on their individual profits. However, there are many firms with different ownership structures. For example, publicly-owned firms tend to maximize the social welfare, but partially publicly-owned firms tend to maximize the weighted average of the social welfare and its own profit (Elsadany, Awad, 2016). The main purpose of this paper is to investigate the dynamic behavior of Cournot oligopoly game incorporating semipublic and private firms where the bounded rational players update their production strategies at discrete time periods by an adjustment mechanism based on maximize their individual profits and the social welfare. This is a partial theoretical approach to our main ongoing research objective, which is to quantify and study an oligopoly of the Greek market. The paper is organized as follows: In Section 2, the dynamics of the Cournot duopoly game with differentiated goods between semi-public and private firms is analyzed. The existence and local stability of the equilibrium points are also analyzed. In Section 3 numerical simulations are used to show complex dynamics via computing Lyapunov numbers, bifurcations diagrams, strange attractors and sensitive dependence on initial conditions. Finally, the paper is concluded in Section 4.

2. THE GAME

2.1 The construction of the game

In this Cournot-type duopoly game there are two firms that produce differentiated goods and offer them at discrete time periods (t = 0, 1, 2, …) on their common market. These two firms take decisions about their production quantities also at discrete-time periods (t = 0, 1, 2, …). In addition, it is considered that two players are homogeneous and more specifically, that both companies choose their productions rationally, following the same adjustment mechanism (bounded rational players). At every discrete period t, each player must form an expectation of the rival's output of the next time period in order to determine the corresponding profit-maximization quantities for the next period t+1. The different consideration in this study is that the first player is a semi-public company and the second one is a private. So, the semi-public company cares about the maximization of percentage combination between the social welfare and its profits and the private company about its own profit-maximization only. It is supposed that q_1, q_2 are the productions of each player, then the inverse demand function (as a function of production quantities) is given by:

$$p_i = \alpha - q_i - dq_j, i, j = 1, 2, i \neq j \tag{1}$$

Also, p_i is the price of i firm's product and α is the positive parameter which expresses the size of the market. So, for the two players it means:

$$p_1 = \alpha - q_1 - dq_2, \quad p_2 = \alpha - q_2 - dq_1 \tag{2}$$

In these equations, $d \in (-1,1)$ is the parameter of the differentiation between the two products. For positive values of the parameter d the larger the value, the less diversification there is between the two products. If $d = 0$, then each company participates in a monopoly game. On the other hand, the negative values of the differentiation parameter describe that two products are complementary.

We assume that both players have linear cost functions:

$$C_i(q_1) = c \cdot q_i \tag{3}$$

which means that for two players the cost functions are the following:

$$C_1(q_1) = c \cdot q_1,\ C_2(q_2) = c \cdot q_2 \tag{4}$$

We use the same positive cost parameter $c > 0$ for two players which is equal to the marginal cost of the players. With these assumptions the profit function for each player is:

$$\Pi_1(q_1, q_2) = p_1 \cdot q_1 - C_1(q_1) = (\alpha - c - q_1 - dq_2) \cdot q_1 \tag{5}$$

and

$$\Pi_2(q_1, q_2) = p_2 \cdot q_2 - C_2(q_2) = (\alpha - c - q_2 - dq_1) \cdot q_2 \tag{6}$$

Then, the marginal profits for the players are given by:

$$\frac{\partial \Pi_1}{\partial q_1} = \alpha - c - dq_2 - 2q_1, \quad \frac{\partial \Pi_2}{\partial q_2} = \alpha - c - dq_1 - 2q_2 \tag{7}$$

The representative consumer maximizes the consumer surplus $CS = U(q_1, q_2) - p_1 q_1 - p_2 q_2$, where the utility function U is assumed to be quadratic: $U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}(q_1^2 + 2dq_1q_2 + q_2^2), a > 0$. Then

$$CS = U(q_1, q_2) - p_1 q_1 - p_2 q_2 = \frac{1}{2}\left(q_1^2 + q_2^2\right) + d \cdot q_1 q_2 \tag{8}$$

The social welfare (W) is given by:

$$W(q_1, q_2) = CS + \Pi_1 + \Pi_2 = -\frac{1}{2}\left(q_1^2 + q_2^2\right) + (\alpha - c) \cdot (q_1 + q_2) - d \cdot q_1 q_2 \infty \tag{9}$$

with marginal welfare:

$$\frac{\partial W}{\partial q_1} = \alpha - c - q_1 - d \cdot q_2 \tag{10}$$

The first player (semi-public firm) cares about the maximization of a function that contains a percentage combination between the social welfare and his profit function. This objective function V is described by the following equation:

$$V(q_1, q_2) = s \cdot W(q_1, q_2) + (1 - s) \cdot \Pi_1(q_1, q_2) \tag{11}$$

where $s \in [0,1]$ is the degree of public ownership. The marginal function of V is given by:

$$\frac{\partial V}{\partial q_1} = s \cdot \frac{\partial W}{\partial q_1} + (1-s) \cdot \frac{\partial \Pi_1}{\partial q_1} \Rightarrow \frac{\partial V}{\partial q_1} = \frac{\partial \Pi_1}{\partial q_1} + s \cdot q_1 \tag{12}$$

Both players follow the same strategy to decide their production quantities (homogeneous players) and they are characterized as bounded rational players. According to the existing literature it means that the first semi-public company decides its production following a mechanism that is described by the equation:

$$\frac{q_1(t+1) - q_1(t)}{q_1(t)} = k \cdot \frac{\partial V}{\partial q_1}, k > 0 \tag{13}$$

and the second player (private company) who is also a bounded rational player follows a similar mechanism that is given by the equation:

$$\frac{q_2(t+1) - q_2(t)}{q_2(t)} = k \cdot \frac{\partial \Pi_2}{\partial q_2} \tag{14}$$

Through this mechanism each player increases his level of adaptation when his marginal utility is positive or decreases his level when his marginal utility is negative, where k is considered as the speed of adjustment for two players. The parameter k is positive (k > 0), and gives the extend variation production of the player is following a given utility signal.

The duopoly's dynamical system is described by:

$$\begin{cases} q_1(t+1) = q_1(t) + k \cdot q_1(t) \cdot \dfrac{\partial V_1}{\partial q_1} \\ q_2(t+1) = q_2(t) + k \cdot q_2(t) \cdot \dfrac{\partial \Pi_2}{\partial q_2} \end{cases} \Leftrightarrow$$

$$\begin{cases} q_1(t+1) = q_1(t) + k \cdot q_1(t) \cdot \left[\alpha - c + (s-2) \cdot q_1(t) - d \cdot q_2(t)\right] \\ q_2(t+1) = q_2(t) + k \cdot q_2(t) \cdot \left[\alpha - c - d \cdot q_1(t) - 2 \cdot q_2(t)\right] \end{cases} \tag{15}$$

We investigate the effect of the parameters: k (speed of adjustment), s (relative profit parameter) and d (differentiation degree) on the dynamics of this system.

2.2 Dynamical analysis

2.2.1 The equilibriums of the game

The equilibrium positions are the nonnegative solutions of the algebraic system:

$$\begin{cases} q_1^* \cdot \dfrac{\partial V}{\partial q_1} = 0 \\ q_2^* \cdot \dfrac{\partial \Pi_2}{\partial q_2} = 0 \end{cases} \tag{16}$$

which obtained by setting: $q_1(t+1) = q_1(t) = q_1^*$ and $q_2(t+1) = q_2(t) = q_2^*$ in the dynamical system of Eq. (15).

- If $q_1^* = q_2^* = 0$, then the equilibrium is $E_0 = (0,0)$.
- If $q_1^* = 0$ and $\dfrac{\partial \Pi_2}{\partial q_2} = 0$, then the equilibrium is $E_1 = \left(0, \dfrac{\alpha - c}{2}\right)$.
- If $q_2^* = 0$ and $\dfrac{\partial V}{\partial q_1} = 0$, then the equilibrium is $E_2 = \left(\dfrac{\alpha - c}{2 - s}, 0\right)$.
- If $\dfrac{\partial V}{\partial q_1} = \dfrac{\partial \Pi_2}{\partial q_2} = 0$, we obtain the system:

$$\begin{cases} \alpha - c + (s-2)q_1^* - dq_2^* = 0 \\ \alpha - c - 2q_2^* - dq_1^* = 0 \end{cases} \tag{17}$$

whose solution is the Nash equilibrium $E_* = \left(\dfrac{(\alpha - c)(2-d)}{2(2-s) - d^2}, \dfrac{(\alpha - c)(2-s-d)}{2(2-s) - d^2} \right)$

if

$$(\alpha - c)(2-d) > 0 \overset{2-d>0}{\Leftrightarrow} \alpha > c \tag{18}$$

$$(\alpha - c)(2-s-d) > 0, \quad 2(2-s) - d^2 > 0 \tag{19}$$

2.2.2 Stability of equilibriums

To study the stability of game's equilibriums, the Jacobian matrix is used. The Jacobian matrix $J(q_1, q_2)$ along the variable strategy (q_1, q_2) is:

$$J(q_1, q_2) = \begin{bmatrix} f_{q_1} & f_{q_2} \\ g_{q_1} & g_{q_2} \end{bmatrix} \tag{20}$$

where:

$$f(q_1,q_2) = q_1 + k \cdot q_1 \cdot \frac{\partial V}{\partial q_1} \tag{21}$$

and

$$g(q_1,q_2) = q_2 + k \cdot q_2 \cdot \frac{\partial \Pi_2}{\partial q_2} \tag{22}$$

The Jacobian matrix is:

$$J(q_1^*, q_2^*) = \begin{bmatrix} 1 + k \cdot \left(\frac{\partial V}{\partial q_1} + q_1^* \cdot \frac{\partial^2 V}{\partial q_1^2} \right) & k \cdot q_1^* \cdot \frac{\partial^2 V}{\partial q_1 \partial q_2} \\ k \cdot q_2^* \cdot \frac{\partial^2 \Pi_2}{\partial q_2 \partial q_1} & 1 + k \cdot \left(\frac{\partial \Pi_2}{\partial q_2} + q_2^* \cdot \frac{\partial^2 \Pi_2}{\partial q_2^2} \right) \end{bmatrix} \tag{23}$$

For E_0 the Jacobian matrix is:

$$J(E_0) = \begin{bmatrix} 1 + k \cdot (\alpha - c) & 0 \\ 0 & 1 + k \cdot (\alpha - c) \end{bmatrix} \overset{A=1+k\cdot(\alpha-c)}{=} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \tag{24}$$

with a double eigenvalue $r = A$. Since $|r| > 1$, E_0 is <u>unstable</u>.
For E_1 the Jacobian matrix is:

$$J(E_1) = \begin{bmatrix} 1 + k \cdot \left[\alpha - c - d \cdot q_2^* \right] & 0 \\ k \cdot q_2^* \cdot \frac{\partial^2 \Pi_2}{\partial q_2 \partial q_1} & 1 - k \cdot (\alpha - c) \end{bmatrix} \underset{C=1-k\cdot(\alpha-c)}{\overset{B=1+k\cdot\left[\alpha-c-d\cdot q_2^*\right]}{=}} \begin{bmatrix} B & 0 \\ D & C \end{bmatrix} \tag{25}$$

with two eigenvalues:

$$r_1 = B \text{ and } r_2 = C$$

Since $r_1 = 1 + k \cdot \frac{(\alpha - c)(2 - d)}{2}$ and $(\alpha - c)(2 - d) > 0$ (Eq.(18)), $|r_1| > 1$ and the equilibrium E_1 is unstable.
For E_2 the Jacobian matrix becomes as:

$$J(E_2)=\begin{bmatrix} 1-k\cdot(\alpha-c) & k\cdot q_1^*\cdot\dfrac{\partial^2 V}{\partial q_1\partial q_2} \\ 0 & 1+k\cdot\left(\alpha-c-d\cdot q_1^*\right)\end{bmatrix} \underset{F=1+k\cdot\left(\alpha-c-d\cdot q_1^*\right)}{\overset{E=1-k\cdot(\alpha-c)}{=}} \begin{bmatrix} E & G \\ 0 & F\end{bmatrix} \tag{26}$$

with two eigenvalues: $r_1=E$ and $r_2=F$.

Since $r_2=1+k\cdot\dfrac{(\alpha-c)(2-s-d)}{2-s}$ and $(\alpha-c)(2-s-d)>0$ (Eq.(19)) , $|r_2|>1$ and the equilibrium E_2 is unstable.

For E_* the Jacobian matrix is:

$$J(E_*)=\begin{bmatrix} 1+k\cdot q_1^*\cdot\dfrac{\partial^2 V}{\partial q_1^2} & k\cdot q_1^*\cdot\dfrac{\partial^2 V}{\partial q_1\partial q_2} \\ k\cdot q_2^*\cdot\dfrac{\partial^2 \Pi_2}{\partial q_2\partial q_1} & 1+k\cdot q_2^*\cdot\dfrac{\partial^2 \Pi_2}{\partial q_2^2}\end{bmatrix} = \begin{bmatrix} 1-k\cdot(2-s)\cdot q_1^* & -k\cdot d\cdot q_1^* \\ -k\cdot d\cdot q_2^* & 1-2k\cdot q_2^*\end{bmatrix} \tag{27}$$

with

$$Tr=2-k\cdot(2-s)\cdot q_1^*-2k\cdot q_2^* \tag{28}$$

and

$$Det=1-k\cdot(2-s)\cdot q_1^*-2k\cdot q_2^*+k^2\cdot\left[2(2-s)-d^2\right]\cdot q_1^*\cdot q_2^* \tag{29}$$

To study the stability of Nash equilibrium we use three conditions that the equilibrium position is locally asymptotically stable when they are satisfied simultaneously (Elaydi, 2005):

$$\begin{array}{ll} \text{(i)} & 1-Det>0 \\ \text{(ii)} & 1-Tr+Det>0 \\ \text{(iii)} & 1+Tr+Det>0 \end{array} \tag{30}$$

From condition (ii) we obtain:

$$1-Tr+Det>0 \Leftrightarrow k^2\left[2(2-s)-d^2\right]\cdot q_1^*\cdot q_2^*>0 \tag{31}$$

and it's always satisfied, because $2(2-s)-d^2>0$.

From the condition (i) we obtain:

$$1-Det>0 \Leftrightarrow (2-s)\cdot q_1^*+2q_2^*-k\left[2(2-s)-d^2\right]\cdot q_1^*\cdot q_2^*>0 \tag{32}$$

Finally, from condition (iii):

$$1+Tr+Det>0 \Leftrightarrow \left[2(2-s)-d^2\right]q_1^*q_2^*\cdot k^2-2\left[(2-s)\cdot q_1^*+2q_2^*\right]\cdot k+4>0 \tag{33}$$

Then, we obtain:

Proposition. *The Nash equilibrium of the discrete dynamical system Eq. (15) is locally asymptotically stable if:*

$$(2-s)\cdot q_1^* + 2q_2^* - k\left[2(2-s)-d^2\right]\cdot q_1^* \cdot q_2^* > 0$$

and

$$\left[2(2-s)-d^2\right]q_1^* q_2^* \cdot k^2 - 2\left[(2-s)\cdot q_1^* + 2q_2^*\right]\cdot k + 4 > 0$$

From Eq. (32):

$$0 < k < \frac{1}{2(2-s)-d^2}\cdot\left(\frac{2-s}{q_2^*}+\frac{2}{q_1^*}\right) \tag{34}$$

(first stability condition for k)

Also, the discriminant of Eq. (33) is positive:

$$\Delta = 4\left[(2-s)\cdot q_1^* + 2q_2^*\right]^2 - 16\left[2(2-s)-d^2\right]q_1^* q_2^* = 4\left[(2-s)\cdot q_1^* - 2q_2^*\right]^2 + 16d^2\cdot q_1^* q_2^* > 0,$$

with $q_1^*, q_2^* \neq 0$

and the condition (iii) is true when:

$$k \in (0, k_1) \cup (k_2, +\infty) \tag{35}$$

(second stability condition for k)

where

$$k_{1,2} = \frac{2\left[(2-s)\cdot q_1^* + 2q_2^*\right] \pm \sqrt{\Delta}}{2\left[2(2-s)-d^2\right]q_1^* q_2^*} \tag{36}$$

are the two real roots of Eq. (33). Then we obtain:

Corollary. *The Nash equilibrium of the discrete dynamical system Eq. (15) is locally asymptotically stable if:*

$0 < k < \frac{1}{2(2-s)-d^2}\cdot\left(\frac{2-s}{q_2^*}+\frac{2}{q_1^*}\right)$ *and* $k \in (0, k_1) \cup (k_2, +\infty)$, *where* $k_{1,2}$ *are the two real roots of Eq. (33).*

3. NUMERICAL SIMULATIONS

3.1 Stability spaces

The three-dimensional stability space (Fig.1) is obtained by the two stability conditions that are described in Proposition, setting specific values for the other parameters $\alpha = 5$ and $c = 1$. We continue with the two-dimensional stability spaces focusing on all combinations of the three basic parameters taking them in pairs. The stability space between the parameters s (horizontal axis) and d (vertical axis) (Figure 2) is obtained by setting: $k = 0.27$. Figure 3 contains the stability space between the parameters s (horizontal axis) and k (vertical axis) for $\alpha = 5,\ c = 1$ and $d = 0.25$. The last stability space (Fig. 4) is between the parameters k (horizontal axis) and d (vertical axis) for $\alpha = 5,\ c = 1$ and $s = 0.5$.

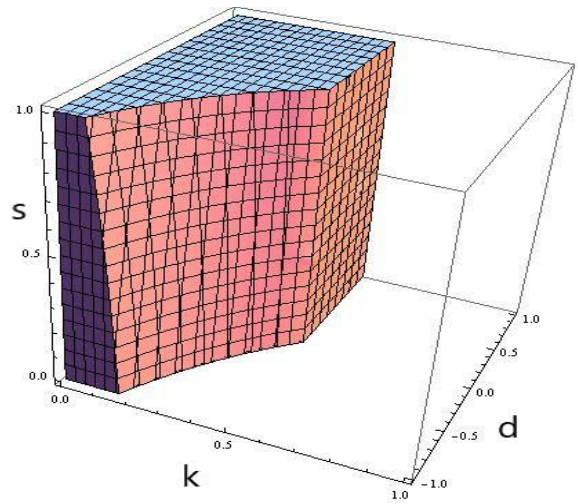

Figure 1: 3D stability space between the parameters k, d and μ for $\alpha = 5,\ c = 1$.

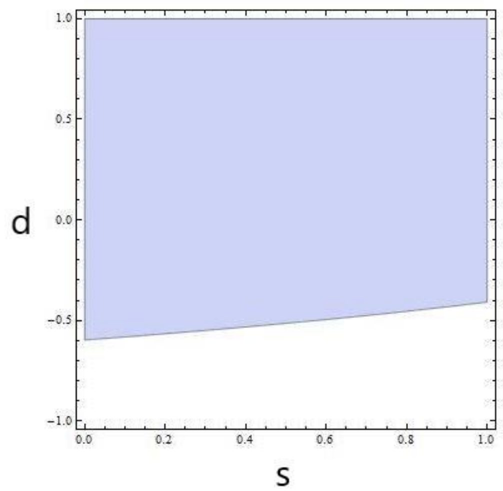

Figure 2: Region of stability between s (horizontal axis) and d (vertical axis) for $\alpha = 5,\ c = 1$ and $k = 0.27$.

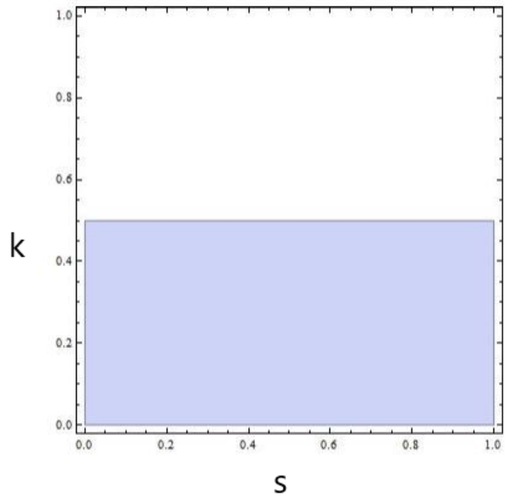

Figure 3: Region of stability between s (horizontal axis) and k (vertical axis) for and $d = 0.25$.

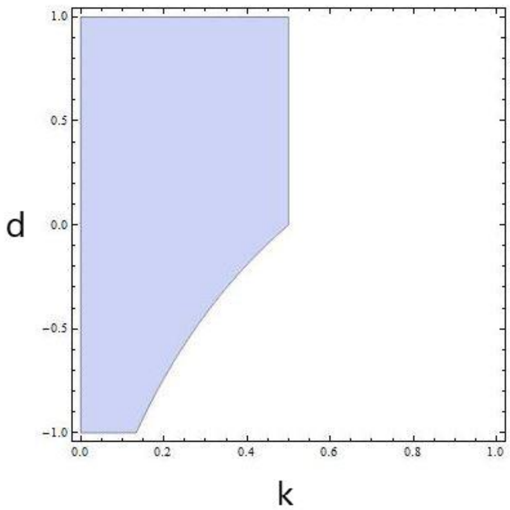

Figure 4: Region of stability between k (horizontal axis) and d (vertical axis) for $\alpha = 5,\ c = 1$ and $s = 0.5$.

3.2 Effect of the parameter k (speed of adjustment)

In this section we present various numerical results focusing on the parameter k, including bifurcation diagrams, strange attractors, Lyapunov numbers and sensitive dependence on initial conditions (Kulenovic, M. and Merino, O.). We choose some fixed values of the others parameters : $\alpha = 5,\ c = 1,\ d = 0.25$ and $s = 0.3$. Then, $q_1^* \approx 2.097$ and $q_2^* \approx 1.737$. The stability conditions for k is : $0 < k < 0.57$ and $k \in (0, 0.49) \cup (0.65, 1)$, or equivalently, $0 < k < 0.49$. It is verified by the bifurcation diagrams of the parameter k against the variables q_1^* (left) and q_2^* (right) that are shown in Fig.5 and Fig.6. These two figures show that the equilibrium undergoes a flip bifurcation at $k = 0.49$. Then a further increase in speed of adjustment implies that a stable two-period cycle emerges for $0.49 < k < 0.60$. As long as the parameter k reduces a four-period cycle, cycles of highly periodicity and a cascade of flip bifurcations that ultimately lead to unpredictable (chaotic) motions are observed when k is larger than 0.64.

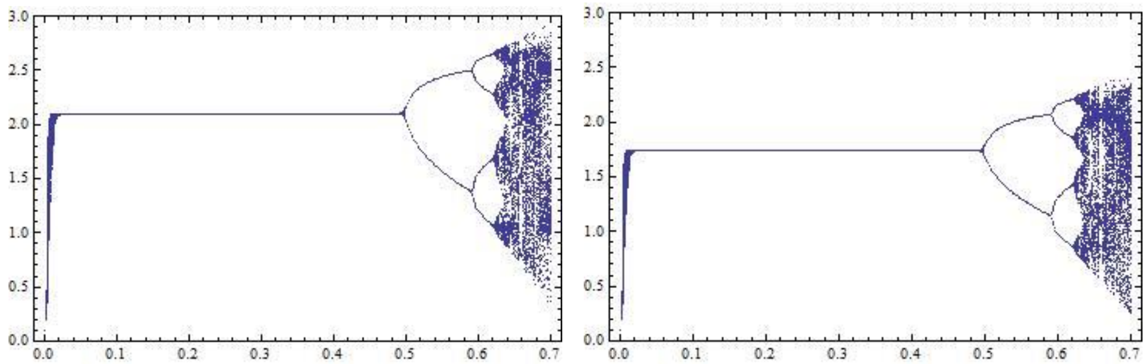

Figure 5: Bifurcation diagrams with respect to the parameter k against the variables q_1^* (left) and q_2^* (right) with 400 iterations of the map Eq.(15) for $\alpha = 5,\ c = 1,\ d = 0.25$ and $s = 0.3$.

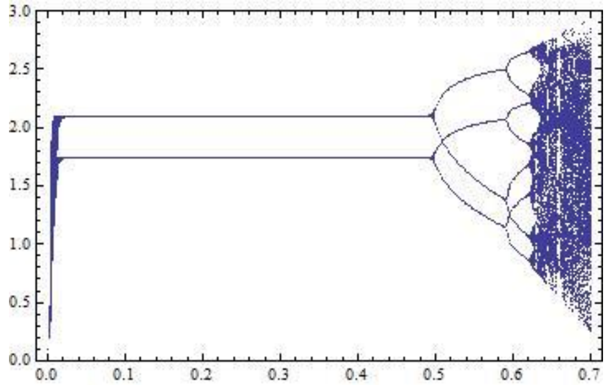

Figure 6: Two bifurcation diagrams of Fig.5 are plotted in one.

This unpredictable (chaotic) behavior of the system Eq. (15) is visualized in Fig. 7 (left) with the strange attractor for $k = 0.7$. This is the graph of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(15) for $\alpha = 5,\ c = 1,\ d = 0.25$ and $s = 0.3$. Also, we use the useful tool of Lyapunov numbers (Fig.7 (right)) (i.e. the natural logarithm of Lyapunov exponents) as a function of the parameter of interest. Figure 7 (right) shows the Lyapunov numbers of the same orbit. It is known that if the Lyapunov number is greater than 1, one has evidence for chaos.

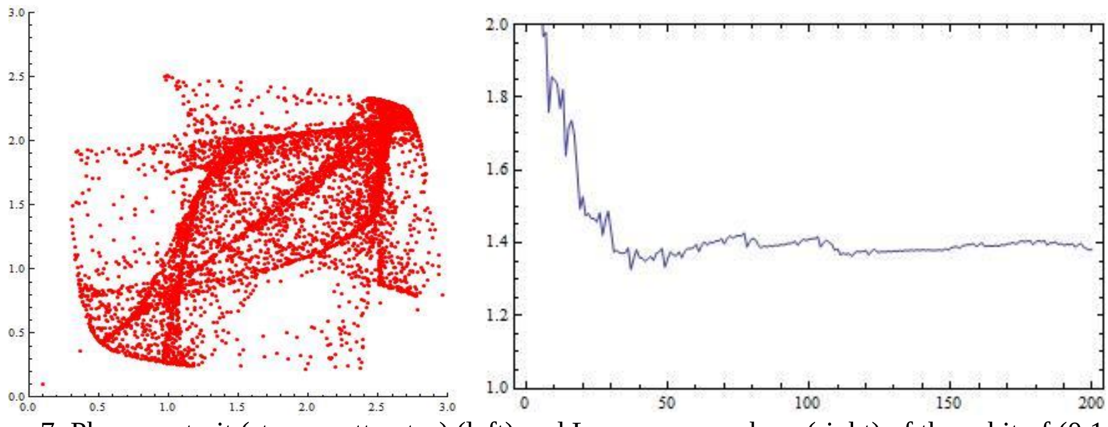

Figure 7: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(15) for $\alpha = 5,\ c = 1,\ d = 0.25,\ s = 0.3$ and $k = 0.7$.

Another characteristic of deterministic chaos is the sensitivity dependence on initial conditions. In order to show the sensitivity dependence on initial conditions of the system Eq.(15), we have computed two orbits with initial points (0.1,0.1) and (0.101,0.1) respectively. Figure 8 shows the sensitivity dependence on initial conditions for q_1- coordinate of the two orbits, for the system

Eq.(15), plotted against the time with the parameter values $\alpha = 5,\ c = 1,\ d = 0.25,\ s = 0.3$ and $k = 0.7$. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly. From these numerical results when all parameters are fixed and only k is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.

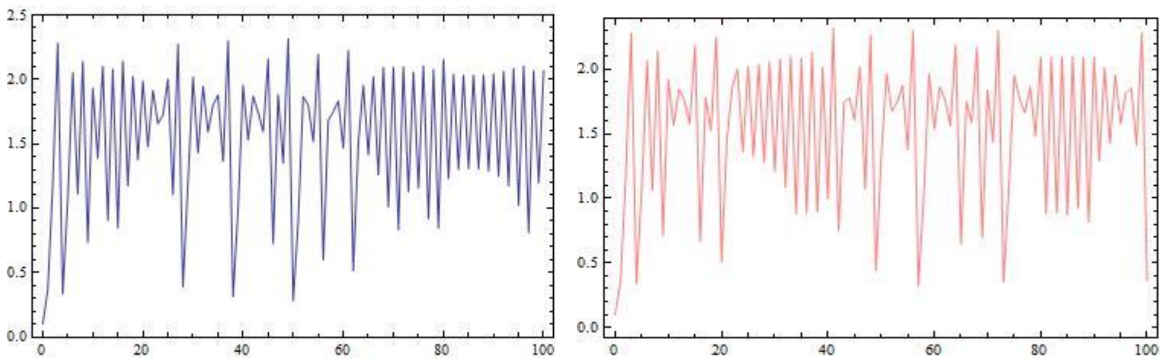

Figure 8: Sensitive dependence on initial conditions for q_1-coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(15) for $\alpha = 5,\ c = 1,\ d = 0.25,\ s = 0.3$ and $k = 0.7$.

3.3 Effect of the parameter d (product differentiation degree)

Using Fig.9 we can find that when $s = 0.6$ and $k = 0.27$ there is a stable equilibrium for $d \in (-0.5, 1)$ and it is verified by the bifurcation diagrams of d against q_1^* (left) and q_2^* (right). Also, a chaotic behavior for the system Eq.(15) appears for negative values of the parameter d (products' differentiation degree) making the system unpredictable. This chaotic behavior can be shown by the strange attractor (Fig.11 (left)) and the Lyapunov numbers (Fig.11 (right)) that are plotted for $d = -0.5$ (outside the stability space). Finally the system Eq.(13) becomes sensitive on small changes of its initial conditions when the parameter d takes small negative values (Fig.12).

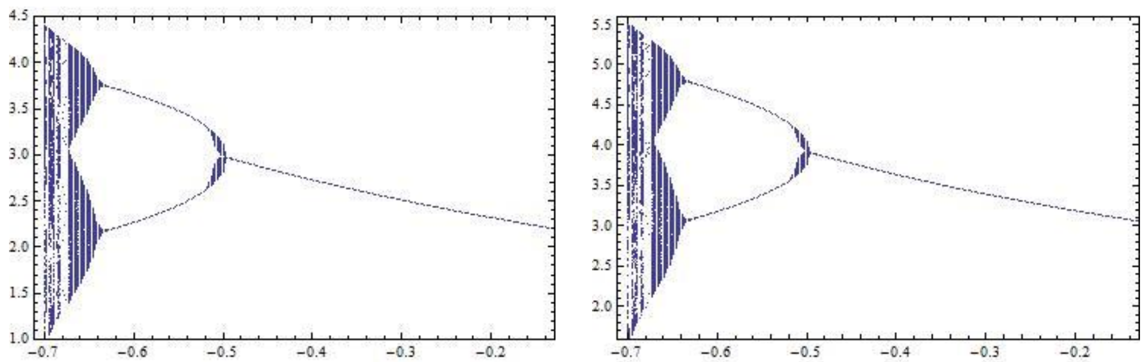

Figure 9: Bifurcation diagrams with respect to the parameter d against the variables q_1^* (left) and q_2^* (right) with 400 iterations of the map Eq.(15) for $\alpha = 5,\ c = 1,\ k = 0.27$ and $s = 0.6$.

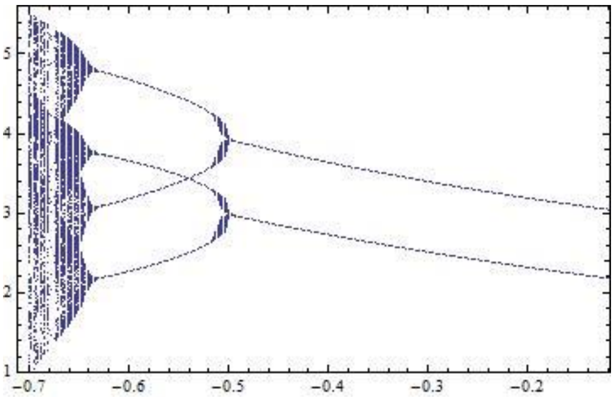

Figure 10: Two bifurcation diagrams of Fig.9 are plotted in one.

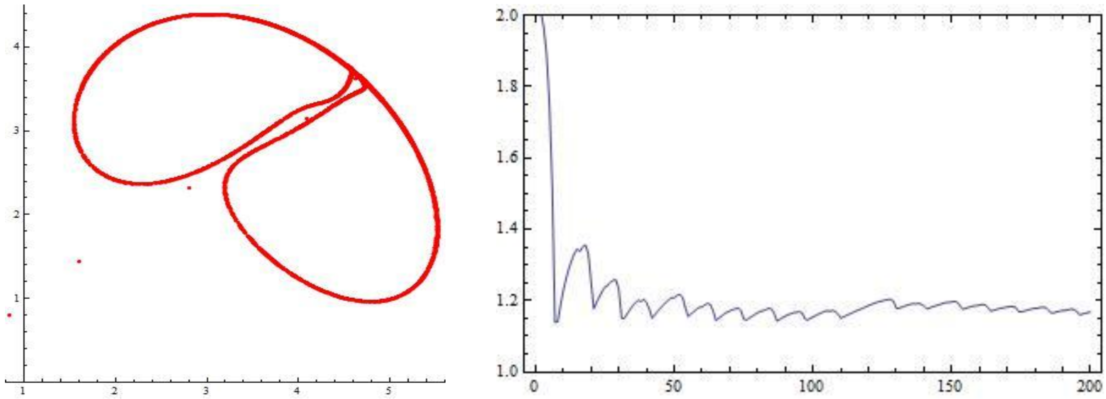

Figure 11: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 2000 iterations of the map Eq.(15) for $\alpha = 5,\ c = 1,\ k = 0.27,\ s = 0.6$ and $d = -0.7$.

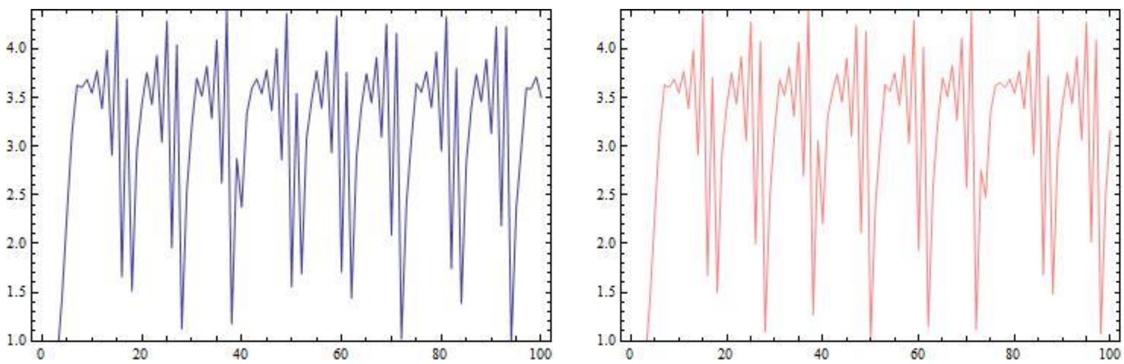

Figure 12: Sensitive dependence on initial conditions for q_1 -coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(15) for $\alpha = 5,\ c = 1,\ k = 0.27,\ s = 0.6$ and $d = -0.7$.

3.4 Effect of the parameter s (the degree of public ownership)

From the stability space between the parameters d and s (Fig.2) it seems that when $d = -0.6$ and $k = 0.27$ there is a stable equilibrium for very small values of the parameter s near zero and it is verified by the bifurcation diagrams of d against q_1^* (left) and q_2^* (right) (Fig.13). Also, a chaotic behavior for the system Eq. (15) appears for large values of the parameter s (relative profit parameter) and a chaotic trajectory appeared. This chaotic behavior can be shown by the strange attractor (Fig.14 (left) and the Lyapunov numbers (Fig.14 (right)) that are plotted for $s = 0.98$. Finally, the system Eq. (15) becomes sensitive on small changes of its initial conditions when the parameter s takes large values (Fig.15). Also, using the stability region of Fig.2 it seems that when k = 0.27 and the parameter d (product's differentiation degree) takes large values, there is a locally asymptotically stable Nash Equilibrium for every value of the parameter s and it means that the parameter s (relative profit parameter) cannot destabilize the economy for specific values of the other parameters α, c and k.

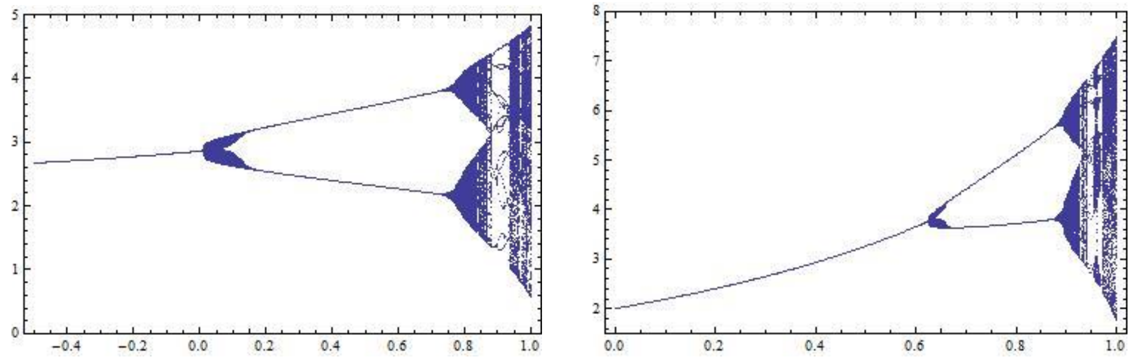

Figure 13: Bifurcation diagrams with respect to the parameter s against the variables q_1^* (left) and q_2^* (right) with 400 iterations of the map Eq.(15) for $\alpha = 5,\ c = 1,\ k = 0.27$ and $d = -0.6$.

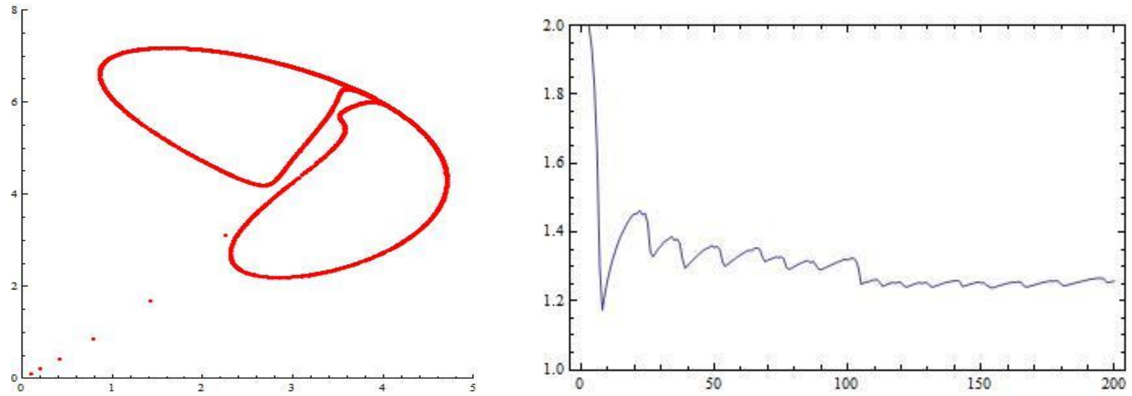

Figure 14: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 2000 iterations of the map Eq.(15) for $\alpha = 5,\ c = 1,\ k = 0.27,\ d = -0.6$ and $s = 0.98$.

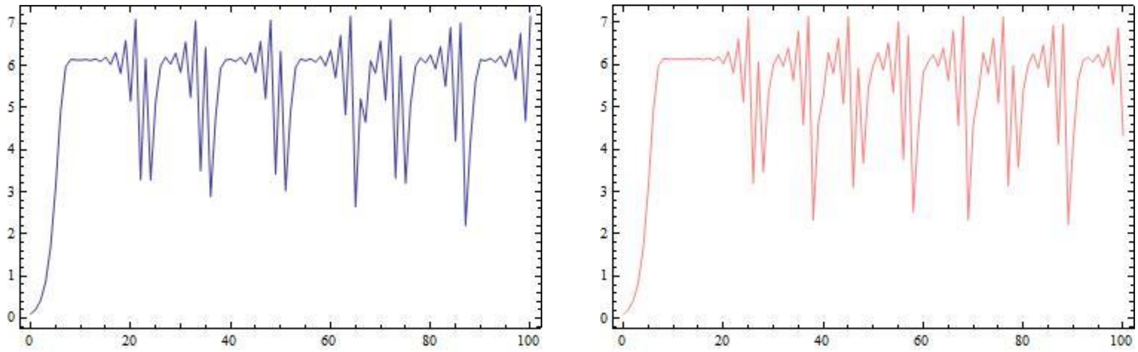

Figure 15: Sensitive dependence on initial conditions for q_1-coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(15) for $\alpha = 5,\ c = 1,\ k = 0.27,\ d = -0.6$ and $s = -0.98$.

4. CONCLUSIONS

The present paper is a partial theoretical approach to our main ongoing research objective, which is to quantify and study an oligopoly of the Greek market. We analyzed through a discrete dynamical system the behavior of a differentiated Cournot duopoly game incorporating semipublic and private firms. By assuming that the bounded rational players update their production strategies at discrete time periods by an adjustment mechanism, based on maximize their individual profits and the social welfare, a discrete dynamic system was obtained. Existence and stability of equilibriums of this system are studied. We showed that the speed of adjustment (k), the parameter of the differentiation (d) and the degree of public ownership (s) may change the stability of equilibrium and cause a structure to behave chaotically. For low values of k and s or for high values of d the Cournot-Nash equilibrium is stable. Increasing (decreasing) these values, the equilibrium becomes unstable, through period doubling bifurcation. Finally, we showed that for lower values of the speed of adjustment the Cournot–Nash equilibrium is stable for each value of the differentiation parameter or the degree of public ownership parameter.

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