NONLINEAR PROGRESSIVE INCOME TAXATION AND INEQUALITIES IN INCOME AND WEALTH BETWEEN HETEROGENEOUS HOUSEHOLDS

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***Abstract**: This paper constructs a dynamic economic growth model in an integrated Walrasian and neoclassical equilibrium theory. This study is concerned with dynamic interactions between progressive nonlinear taxation and wealth and income distributions between heterogeneous households. The economy is composed of one industrial sector and public sector with labor and capital as input factors. The government taxes households and the tax income is entirely spent on supply public services. The population is classified into any number of households and households use disposable incomes for consumption and saving. The machine of economic growth is wealth accumulation. The model builds dynamic interdependence between economic growth, economic structural change, capital/wealth accumulation, progressive nonlinear income taxation, and public services. We simulate the model to demonstrate dynamic properties of the economic system. We show the existence of equilibrium and follow the motion of the dynamic system. We conduct comparative dynamic analysis in different parameters.*

Keywords: Progressive nonlinear taxation; Wealth accumulation; Walrasian theory; Economic structure; Income and wealth distribution; Public services.

JEL Classification Codes: D31, D51, O41.

1. INTRODUCTION

This study examines a significant topic in economics in a general equilibrium framework. In most of formal growth models, taxation is often assumed to a proportion of income or output. Nevertheless, it is often argued that government taxes agents in a not fixed proportion but in complicated relation to income levels of different households and national economic growth. Empirical studies demonstrate that developed economies are characterized of "progressive income taxation together with utility-generating public spending" (Chen and Guo, 2014: 174). As with regard to many empirically observed reality, the history of analytical economics shows that it is not theoretically easy to explain economic growth with heterogeneous households, even without taxation and impact of tax income on utility levels. As shown in Zhang (2005, 2008), the lacking of proper analytical framework in the mainstreams of economic theory explains why economics has failed to provide proper insights into issues related with growth and distribution under various policies. Zhang (1993, 2008) has made attempts to construct an alternative approach to economic growth. This study applies Zhang's analytical framework to deal issues related to growth, distribution, and "progressive income taxation together with utility-generating public spending". It should be noted that there are studies on dynamic relationship between government's spending and private consumption in a dynamic general equilibrium framework (Barro, 1990; Glomm and

Ravikumar, 1994, 1997; Turnovsky, 2000, 2004; Palivos *et al*., 2003; Greiner, 2007; Agénor, 2011). We deal with the issues differently from the traditional Ramsey approaches.

We are focused on dynamics of income and wealth distribution with different taxation policies on heterogeneous households. In the literature of economic growth theory, there only a few studies with microeconomic foundation and heterogenous households (e.g., Burmeister and Dobell, 1970; Jones and Manuelli, 1997). We analyze growth and inequality with progressive income taxation within an integrated framework of the Walrasian general equilibrium and neoclassical growth theories by Zhang. Although the Walrasian theory is mathematically well developed, it is not effective to take account of wealth accumulation (e.g., Morishima, 1964, 1977; Impicciatore *et al*., 2012). It should be also mentioned that this study is based on a model by Zhang (2016). This study differs from Zhang's model mainly in that this study is developed for a closed economy, while Zhang's model is for a small open economy. The rest of this paper is organized as follows. Section 2 develops a heterogeneous-households model with nonlinear progressive taxation. Section 3 examines dynamic properties of the model, providing a computational procedure for simulating the motion and simulating a three-group economy. Section 4 conducts comparative dynamic analysis in some parameters. Section 5 concludes the study.

2. THE BASIC MODEL

This section constructs a two-sector neoclassical growth model with heterogeneous households and progressive taxation. The economy consists of industrial and public sector. The industrial sector is the same as in Solow's one-sector growth model. It produces a single good which is used for consumption and investment. The public sector provides public services, which are consumed by households. All prices are measured with that of industrial good. The price of commodity is unity. There are J groups of households, indexed by $j = 1, ..., J$. Group $j's$ We denote the rate of interest by $r(t)$. Capital is depreciated by a constant exponential rate δ_k. Most aspects of the industrial sector are based on neoclassical growth theory (Burmeister and Dobell, 1970; Barro and Sala-i-Martin, 1995; and Zhang, 2005). Each worker is employed in either of the two sectors. Assets of the economy are owned privately. Households use up their incomes for consuming and saving. Two sectors fully employ labor and capital inputs available in markets. Exchanges take place in perfectly competitive markets. Factor markets work well and factors are fully utilized at every moment. Saving is undertaken only by households. Labor force and capital stock are distributed between the two sectors. The labor force of national economy is

$$N = \sum_{j=1}^{J} h_j N_j , \qquad (1)$$

where h_j is the level of human capital of group j. This study assumes h_j to be fixed.

The industrial sector

We specify the production function as follows:

$$F(t) = A_i\, K_i^{\alpha_i}(t) N_i^{\beta_i}(t), \;\; \alpha_i + \beta_i = 1, \;\; \alpha_i , \beta_i > 0, \qquad (2)$$

where A_i, α_i and β_i are parameters. The marginal conditions are:

$$r(t) = \frac{\alpha_i\, F(t)}{K_i(t)} - \delta_k , \;\; w(t) = \frac{\beta_i F(t)}{N_i(t)}, \;\; w_j(t) = h_j\, w(t). \qquad (3)$$

The current and disposable income

We use Zhang's approach to model behaviour of households (Zhang, 1993, 2005). We use $\bar{k}_j(t)$ to represent the value of wealth owned by the representative household of group j.

When household j does not pay any tax on the current income, the household income from the interest payment and the wage payment is:

$$y_{0j}(t) = r(t)\bar{k}_j(t) + w_j(t). \qquad (4)$$

On the basis of Chen and Guo (2014) and Zhang (2016), we introduce the progressive tax rate $\tau_j(t)$ as a function of $y_{0j}(t)$ as follows:

$$\tau_j(t) = \tau_{0j} + b_j\, y_{0j}^{a_j}(t), \;\; 1 > \tau_{0j}, \, a_j > 0, \, b_j > 0. \qquad (5)$$

The condition of $a_j > 0$ implies that tax rate is increased as per capita income is increased if b_j . is positive. The tax schedule is said to be progressive. In the case of $b_j = 0$ the tax schedule is called flat. It is commonly assumed that tax rate on income is constant or flat (e.g., Cazzavillan, 1996; Raurich, 2003; Fernández *et al*. 2004; Chen, 2006; Guo and Harrison, 2008). Household $j's$ current income $y_j(t)$ under the taxation is

$$y_j(t) = \bar{\tau}_j(t)\, y_{0j}(t), \qquad (6)$$

where $\bar{\tau}_j(t) = 1 - \tau_j(t)$. It is assumed that selling and buying wealth can be conducted instantaneously without any transaction cost. The disposable income is the sum of the current income and the value of wealth owned by the household as follows:

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t) = (1 + \bar{\tau}_j(t) r(t))\bar{k}_j(t) + \bar{\tau}_j(t) w_j(t). \qquad (7)$$

The budget constraints

The disposable income is expended entirely on saving and consuming. We have budget constraint as follows:

$$c_j(t) + s_j(t) = \hat{y}_j(t). \qquad (8)$$

Utility functions and optimal decisions

The representative household chooses consumption and saving subject to the budget constraint. We assume that utility level $U_j(t)$ is dependent on consumption of commodity and saving as follows:

$$U_j(t) = u_j(G(t),t)c_j^{\xi_{0j}}(t)s_j^{\lambda_{0j}}(t), \;\; \xi_{0j}, \, \lambda_{0j} > 0,$$

where u_j is dependent on the level of public services $G(t)$, and ξ_{0j} and λ_{0j} are respectively the propensities to consume good and to hold wealth.

The household's optimal behaviour

Household j maximizes U_j subject to budget constraint (8). The marginal conditions are as follows:

$$c_j(t) = \xi_j\, \hat{y}_j(t), \;\; s_j(t) = \lambda_j\, \hat{y}_j(t), \qquad (9)$$

where

$$\rho_j \equiv \frac{1}{\xi_{0j} + \lambda_{0j}}, \;\; \xi_j \equiv \rho_j\, \xi_{0j}, \;\; \lambda_j \equiv \rho_j\, \lambda_{0j}.$$

Wealth accumulation

According to the definitions of $s_j(t)$ and $\bar{k}_j(t)$, the change in wealth is saving minus dissaving as follows:

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t). \qquad (10)$$

The public sector

Following Zhang (2015), we model the public sector. The public sector is financially supported by the tax income. The public sector pays capital stocks and workers at the same rates that the private sector pays these input factors. The public sector's tax income is given as follows:

$$I_p(t) = \sum_{j=11}^{J} \left[r(t)\bar{k}_j(t) + w_j(t)\right]\tau_j(t) N_j . \qquad (11)$$

We assume that the public sector is efficient in the sense that it optimally uses its resource to maximize public services. Public services is a function of capital $K_p(t)$ and labor force $N_p(t)$ as follows:

$$G(t) = A_p K_p^{\alpha_p}(t) N_p^{\beta_p}(t), \quad A_p, \alpha_p, \beta_p > 0, \quad \alpha_p + \beta_p = 1.$$

The budget constraint of the public sector is

$$w(t)N_p(t) + (r(t) + \delta_k)K_p(t) = I_p(t). \qquad (12)$$

The public sector behaves effectively in the sense that it uses the available resource to maximize public services. The marginal conditions imply

$$(r(t) + \delta_k)K_p(t) = \alpha_p I_p(t), \quad w(t)N_p(t) = \beta_p I_p(t). \qquad (13)$$

The input factors are fully employed

The national capital stock and the labor force are fully employed

$$K_i(t) + K_p(t) = K(t), \quad N_i(t) + N_p(t) = N. \qquad (14)$$

The national wealth equaling the value of physical capital

The national wealth $\bar{K}(t)$ is equal to the sum of the wealth owned by all the households in the country

$$\bar{K}(t) = \sum_{j=1}^{J} \bar{k}_j(t) N_j . \qquad (15)$$

The national wealth is equal to $K(t)$

$$K(t) = \bar{K}(t). \qquad (16)$$

We constructed the two-sector model with heterogeneous households and progressive income taxation. As the system has many types of households, the dynamics is composed of high dimension. The next section studies properties of the economic system.

3. DYNAMIC BEHAVIOR OF THE MODEL

This section examines properties of the dynamic system. We now show that we can follow the dynamics by a set of nonlinear differential equations. The dimension is equal to the number of types of households. First, we introduce variables

$$z_1(t) \equiv \frac{r(t) + \delta_k}{w(t)}. \quad \{\bar{k}_j(t)\} \equiv (\bar{k}_2(t), \ldots, \bar{k}_J(t)).$$

Lemma

With $a_1 = 1$, we have that the dynamics of the economic system is governed by the following J differential equations

$$\dot{z}(t) = \Lambda_1(z(t), \{\bar{k}_j(t)\}),$$
$$\dot{\bar{k}}_j(t) = \Lambda_j(z(t), \{\bar{k}_j(t)\}), \quad j = 2, \ldots, J, \qquad (17)$$

where functions Λ_j are defined in the Appendix. All the variables are functions of $z(t)$ and $\{\bar{k}_j(t)\}$ as follows: $r(t)$ with (A3) $\rightarrow$ $w(t)$ from (A3) $\rightarrow w_j(t) = h_j\, w(t)$ $\rightarrow \bar{k}_1(t)$ from (A9) $\rightarrow$ $y_{0j}(t)$ by (4) $\rightarrow$ $\tau_j(t)$ by (5) $\rightarrow$ $\hat{y}_j(t)$ from (7) $\rightarrow$ $c_j(t)$ and $s_j(t)$ by (8) $\rightarrow$ $I_p(t)$ by (11) $\rightarrow$ $\overline{K}(t)$ by (15) $\rightarrow$ $K(t)$ by (A16) $\rightarrow$ $N_p(t)$ and $K_p(t)$ by (13) $\rightarrow$ $N_i(t)$ and $K_i(t)$ by (14) $\rightarrow$ $F_i(t)$ by (A2) $\rightarrow G(t)$ by the definition.

As the dynamic system is complicated, we cannot analyze analytical properties of the nonlinear dynamic system. As we have a computational procedure, it is straightforward for us to simulate the model. We simulate an economy 3 groups of households. We specify that different households benefit from public services differently as follows:

$$u_1(t) = 0.5G^{0.1}(t), \; u_2(t) = 0.5G^{0.15}(t), \; u_3(t) = 0.5G^{0.2}(t).$$

This implies that group 3's marginal utility of public goods is higher than group 2's and group 2's marginal utility of public goods is higher than group 1's. We specify the parameter values as follows:

$$N_1 = 10,\; N_2 = 30,\; N_3 = 60,\; T_0 = 1,\; \alpha_i = 0.32,\; \alpha_p = 0.4,\; A_i = 1.5,\; A_p = 0.9,$$
$$h_1 = 12,\; h_2 = 4,\; h_3 = 2,\; \xi_{01} = 0.15,\; \lambda_{01} = 0.7,\; \xi_{02} = 0.18,\; \lambda_{02} = 0.7,\; \xi_{03} = 0.2,$$
$$\lambda_{03} = 0.6,\; \tau_{01} = 0.03,\; b_1 = 0.01,\; a_1 = 1,\; \tau_{02} = 0.03,\; b_2 = 0.01,\; a_2 = 0.8,$$
$$\tau_{03} = 0.03,\; b_3 = 0.01,\; a_3 = 0.3,\; \delta_k = 0.05. \qquad (18)$$

Group 1 has highest propensity to save and highest level of human capital; group 2's are the next, and group 3's the lowest. The total productivities of the two sectors are specified at 1.5 and 0.9, respectively. As in some empirical studies we specify the value of the parameter, α, in the Cobb-Douglas production is approximately 0.3 (e.g., Miles and Scott, 2005, Abel *et al*, 2007). We start the economy with the following initial conditions:

$$z(0) = 0.065,\; \bar{k}_2(0) = 36,\; \bar{k}_3(0) = 13.$$

The movement of the economic system is plotted in Figure 1. The tax rates change slightly over time. The rate of interest falls. Some of the labor force is moved from the industrial sector to the public sector. The two sectors expand. The capital stocks and national wealth are increased over time. Group 1's consumption, wealth and disposable income rise, while the other two groups' consumption levels, wealth and disposable incomes change slightly. Group 1's utility level rises, while the other two groups' utility levels change slightly.

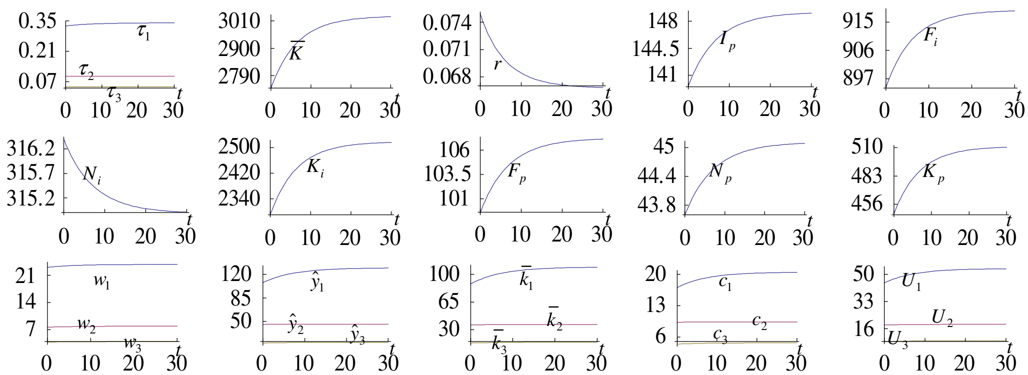

Figure 1. The Motion of the Dynamic System

As shown in Figure 1, the system becomes stationary in the long term. The equilibrium values of the variables are given as follows:

$$r = 0.067,\ w_1 = 23.8,\ w_2 = 7.94,\ w_3 = 3.97,\ \bar{K} = 3030.5,\ \tau_1 = 0.341,$$
$$\tau_2 = 0.095,\ \tau_3 = 0.046,\ I_p = 149.2,\ F_i = 918.8,\ F_p = 107.2,\ N_i = 314.9,$$
$$N_p = 45.1,\ K_i = 2519.2,\ K_p = 511.3,\ \hat{y}_1 = 129.8,\ \hat{y}_2 = 45.9,\ \hat{y}_3 = 18.7,$$
$$\bar{k}_1 = 109.3,\ \bar{k}_2 = 36.5,\ \bar{k}_3 = 14,\ c_1 = 20.5,\ c_2 = 9.4,\ c_3 = 4.7,\ U_1 = 53.7,$$
$$U_2 = 18.7,\ U_3 = 8.46.$$

We calculate the three eigenvalues as follows:

$$-0.2,\ -0.16,\ -0.14.$$

The equilibrium point is locally stable. This result guarantees that we can effectively conduct comparative dynamic analysis in the next section.

4. COMPARATIVE DYNAMIC ANALYSIS

Comparative dynamic analysis shows how changes in exogenous conditions affect processes of changes in the system variables. In most of theoretical dynamic models only comparative static analysis is conducted as the economic systems are unstable. As we gave the computation procedure to simulate the system, it is straightforward to conduct comparative dynamic analysis. We introduce a symbol $\bar{\Delta}$ stands for the change rate of the variable in percentage due to changes in the parameter value.

4.1. The constant parts in the three groups' tax rates are increased

We first study what will happen to the economic system if the government strengthens its taxation in the following way:

$$\tau_{01}: 0.03 \Rightarrow 0.032,\ \tau_{02}: 0.03 \Rightarrow 0.032,\ \tau_{03}: 0.03 \Rightarrow 0.032.$$

The result is plotted in Figure 2. We see that the tax rates on the three groups are increased. The national wealth falls and the rate of interest rises. The public sector expands, but the industrial sector shrinks. The wage rates fall. Except very short period, all the groups have lower levels of disposable incomes, less wealth, lower consumption levels, and lower utility levels. We see that the strengthened taxation makes no group better off, except expanding the public sector.

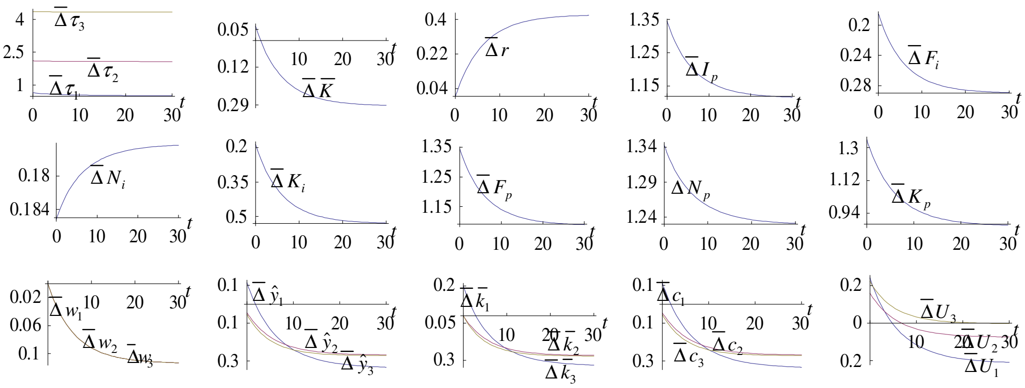

Figure 2. The Government Strengthens Taxation on the Three Groups

4.2. Groups 2 and 3 enhance human capital

We first study what will happen to the economic system if the government strengthens its taxation in the following way:

$$h_2 : 4 \Rightarrow 4.5, \quad h_3 : 2 \Rightarrow 2.5.$$

The result is plotted in Figure 3. Group 1's tax rate rises initially and changes slightly in the long term. Groups 2 and 3' tax rates are enhanced. The two sectors expand. The national wealth rises. The rate of interest falls in initially and rises in the long term. The total tax income is increased. Group 2 and 3's utility levels, disposable incomes, consumption and per household wealth are all increased. Group 1's economic conditions are slightly affected in the long term.

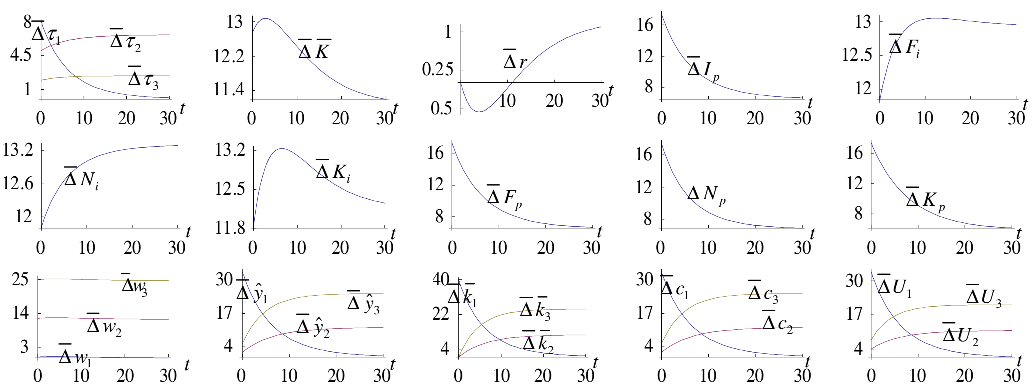

Figure 3. Groups 2 and 3 Enhance Human Capital

4.3. Groups 2 and 3 enhance the propensities to save

We now analyze the impact of the following rises in groups 2 and 3's propensities to save on the motion of the economic system:

$$\lambda_{02} : 0.7 \Rightarrow 0.72, \quad \lambda_{03} : 0.6 \Rightarrow 0.62.$$

The result is plotted in Figure 4. Group 1's tax rate falls. Groups 2 and 3' tax rates are enhanced. The two sectors expand. The national wealth rises. The rate of interest falls. The wage rate are augmented. The total tax income is increased. Group 2 and 3's utility levels, disposable incomes, and per household wealth are all increased. Group 1's economic conditions are slightly affected in the long term.

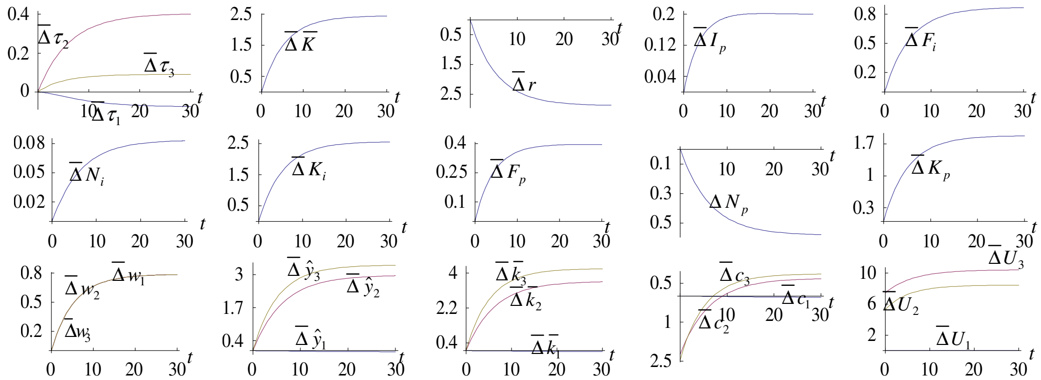

Figure 4. Groups 2 and 3 Enhance Their Propensities to Save

4.4. The tax rates are more strongly related to disposable incomes

We now show what will happen to the economic dynamics when the tax rates are more strongly related to disposable incomes as follows:

$$b_j : 0.01 \Rightarrow 0.012, \quad j = 1, 2, 3.$$

The result is plotted in Figure 5. All the tax rates are augmented. The national wealth falls in the long term. The industrial sector produces and employs less input factors. The tax income is increased. The public sector supplies more public services and employs more input factors. The rate of interest rises. The wage rates are reduced. All the groups have lower disposable incomes, lower consumption levels, and less wealth. Group 1 has lower utility level, while the other two groups enjoy higher utilities. As group 1's economic conditions lose more than the other two groups, we see that inequalities between group 1 and the other two groups are reduced.

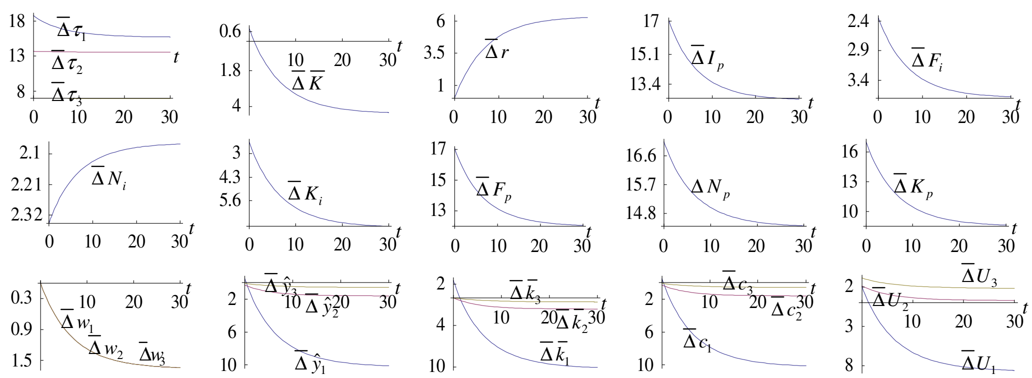

Figure 5. The Tax Rates Are More Strongly Related to Disposable Incomes

4.5. The industrial sector's total factor productivity is enhanced

We allow the industrial sector's total factor productivity to be augmented as follows:

$$A_i : 1.5 \Rightarrow 1.55.$$

The result is plotted in Figure 6. All the tax rates are augmented. The national wealth rises. The rate of interest is increased. The government has more income. The public sector employs more labor force, while the industrial sector employs less labor force. The two sectors employ more capital goods. The two sectors expand. The wage rates are increased. All the groups have higher disposable incomes, higher consumption levels, more wealth, and higher utility levels.

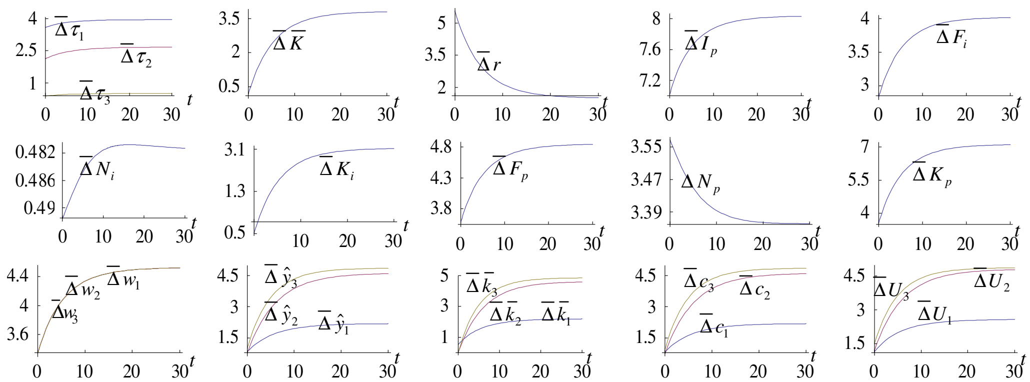

Figure 6. The Industrial Sector's Total Factor Productivity Is Enhanced

5. CONCLUSIONS

This paper constructed a dynamic economic growth model in an integrated Walrasian and neoclassical equilibrium theory. The paper was concerned with dynamic interactions between progressive nonlinear taxation and wealth and income distributions between heterogeneous households. The economy consists of one industrial sector and public sector. The two sectors use labor and capital as input factors. The factor markets are perfectly competitive. The government taxes households and the tax income is entirely spent on supply public services. The population is classified into any number of households and households use disposable incomes for consumption and saving. The model built dynamic interdependence between economic growth, economic structural change, capital/wealth accumulation, progressive nonlinear income taxation, and public services. We simulated the model and confirmed dynamic properties of the economic system. We also showed the existence of equilibrium and followed the motion of the dynamic system. We conducted comparative dynamic analysis in groups 2 and 3's human capital, groups 2 and 3's propensities to save, proportional parts of the tax rates, and the industrial sector's total factor productivity. We show how changes in these parameters affect the national output, the two sectors, the households' consumption and saving levels, and distribution between income and wealth in transitory processes as well as the long term steady state. We may generalize the model in different ways. It is possible to extend the model for any number of national economies. We may also study a multi-sector economy. It is possible to examine effects of other forms of taxation.

APPENDIX: PROVING THE LEMMA

We now confirm the Lemma. From (3) and (13) we have

$$z \equiv \frac{r + \delta_k}{w} = \frac{N_i}{\bar{\beta}_i K_i} = \frac{N_p}{\bar{\beta}_p K_p}, \tag{A1}$$

where $\bar{\beta}_j \equiv \beta_j / \alpha_j$, $j = i, p$. From (A1) and (2), we get

$$F_i = A_i \bar{\beta}_i^{\beta_i} K_i z^{\beta_i}. \tag{A2}$$

With (3) and (A2), we get

$$r = \alpha_i A_i \bar{\beta}_i^{\beta_i} z^{\beta_i} - \delta_k, \quad w = \frac{r + \delta_k}{z}. \tag{A3}$$

From (A1) and (14)

$$\bar{\beta}_i K + \left(\bar{\beta}_p - \bar{\beta}_i\right) K_p = \frac{N}{z}, \tag{A4}$$

where we also apply $K_i = K - K_p$. Insert (11) in (13)

$$K_p = \frac{\alpha_p}{r + \delta_k} \sum_{j=1}^{J} \left(r \bar{k}_j + w_j\right) \tau_j N_j . \qquad \text{(A5)}$$

Insert (A5) in (A4)

$$\bar{\beta}_i z K + \bar{w} \sum_{j=1}^{J} \left(r \bar{k}_j + w_j\right) \tau_j N_j = N , \qquad \text{(A6)}$$

where

$$\bar{w} \equiv \frac{\left(\bar{\beta}_p - \bar{\beta}_i\right) \alpha_p}{w}.$$

Insert (15) in (A6)

$$\bar{\beta}_i z \bar{k}_1 + r \bar{w} \, \tilde{w} \bar{k}_1 + \bar{w} w_1 \tilde{w} + \left(r \bar{w} \bar{k}_1 + \bar{w} w_1\right) b_1 r \bar{k}_1 = n , \qquad \text{(A7)}$$

where we use the definition of y_{01} and the condition $a_{01} = 1$ and

$$\tilde{w} \equiv \tau_{01} + b_1 w_1 , \quad n \equiv \left(N - \bar{\beta}_i z \sum_{j=2}^{J} \bar{k}_j N_j - \bar{w} \sum_{j=2}^{J} \left(r \bar{k}_j + w_j\right) \tau_j N_j \right) \frac{1}{N_1}.$$

From (A7), we have

$$\bar{k}_1^2 + m_1 \bar{k}_1 + m_2 = 0 , \qquad \text{(A8)}$$

where

$$m_1 \equiv \frac{\bar{\beta}_i z + r \bar{w} \, \tilde{w} + b_1 r \bar{w} w_1}{b_1 r r \bar{w}}, \quad m_2 \equiv \frac{\bar{w} w_1 \tilde{w} - n}{b_1 r r \bar{w}}.$$

Solve (A8)

$$\bar{k}_1 = \varphi\left(z, \{\bar{k}_j\}\right) \equiv \frac{- m_1 \pm \sqrt{m_1^2 - 4 m_2}}{2}. \qquad \text{(A9)}$$

In our simulation the following solution of (A10) has a meaningful solution

$$\bar{k}_1 = \frac{- m_1 - \sqrt{m_1^2 - 4 m_2}}{2}.$$

We can see that the variables can be expressed as functions of z and $\{\bar{k}_j\}$ as follows: r with (A3) $\rightarrow$ w from (A3) $\rightarrow w_j = h_j w$ $\rightarrow \bar{k}_1$ from (A9) $\rightarrow$ y_{0j} by (4) $\rightarrow$ τ_j by (5) $\rightarrow$ $\hat{y}_j$ from (7) $\rightarrow$ c_j and s_j by (8) $\rightarrow I_p$ by (11) $\rightarrow$ $\bar{K}$ by (15) $\rightarrow$ K by (A16) $\rightarrow$ N_p and K_p by (13) $\rightarrow$ N_i and K_i by (14) $\rightarrow$ F by (A2) $\rightarrow G$ by the definition. From this procedure and (10), we have

$$\dot{\bar{k}}_1 = \Omega\left(z, \{\bar{k}_j\}\right) \equiv \lambda_1 \hat{y}_1 - \bar{k}_1 , \qquad \text{(A10)}$$

$$\dot{\bar{k}}_j = \Lambda_j\left(z, \{\bar{k}_j\}\right) \equiv \lambda_j \hat{y}_j - \bar{k}_j , \quad j = 2, ..., J . \qquad \text{(A11)}$$

Taking derivatives of (A9) in t and using (A11), we get

$$\dot{\bar{k}}_1 = \frac{\partial \varphi}{\partial z} \dot{z} + \sum_{j=2}^{J} \Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j}. \qquad \text{(A12)}$$

From (A11) and (A12), we get

$$\dot{z} = \Lambda_1\left(z, \{\bar{k}_j\}\right) \equiv \left[\Omega - \sum_{j=2}^{J} \Lambda_j \frac{\partial \varphi}{\partial \bar{k}_j}\right] \left(\frac{\partial \varphi}{\partial z}\right)^{-1}. \qquad \text{(A13)}$$

In summary, we got the Lemma.

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