# ON A COURNOT DUOPOLY GAME WITH DIFFERENTIATED GOODS, HETEROGENEOUS EXPECTATIONS AND A COST FUNCTION INCLUDING EMISSION COSTS

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***Abstract:*** *In this study we investigate the dynamics of a nonlinear Cournot- type duopoly game with differentiated goods, linear demand and a cost function that includes emission costs. The game is modeled with a system of two difference equations. Existence and stability of equilibria of this system are studied. We show that the model gives more complex chaotic and unpredictable trajectories as a consequence of change in the parameter of horizontal product differentiation and a higher (lower) degree of product differentiation (weaker or fiercer competition) destabilize (stabilize) the economy. The chaotic features are justified numerically via computing Lyapunov numbers and sensitive dependence on initial conditions. Also, we show that in this case there are stable trajectories and a higher (lower) degree of product differentiation does not tend to destabilize the economy.*

**Key words**: Cournot duopoly game; Discrete dynamical system; Heterogeneous expectations; Stability; Chaotic behavior.

**JEL Classification Codes : C62, C72, D43.**

## 1. INTRODUCTION

An Oligopoly is a market structure between monopoly and perfect competition, where there are only a few number of firms in the market producing homogeneous products. The dynamic of an oligopoly game is more complex because firms must consider not only the behaviors of the consumers, but also the reactions of the competitors i.e. they form expectations concerning how their rivals will act. Cournot, in 1838 has introduced the first formal theory of oligopoly. He treated the case with naive expectations, so that in every step each player (firm) assumes the last values that were taken by the competitors without estimation of their future reactions.

Expectations play an important role in modelling economic phenomena. A producer can choose his expectations rules of many available techniques to adjust his production outputs. In this paper we study the dynamics of a duopoly model where each firm behaves with heterogeneous expectations strategies. We consider a duopoly model where each player forms a strategy in order to compute his expected output. Each player adjusts his outputs towards the profit maximizing amount as target by using his expectations rule. Some authors considered duopolies with homogeneous expectations and found a variety of complex dynamics in their games, such as appearance of strange attractors (Agiza, 1999, Agiza et al., 2002, Agliari et al., 2005, 2006, Bischi, Kopel, 2001, Kopel, 1996, Puu, 1998, Sarafopoulos, 2015). Also models with heterogeneous agents were studied (Agiza, Elsadany , 2003, 2004, Agiza et al., 2002, Den Haan , 20013, Fanti, Gori, 2012, Tramontana , 2010, Zhang , 2007).

In the real market producers do not know the entire demand function, though it is possible that they have a perfect knowledge of technology, represented by the cost function. Hence, it is more likely that firms employ some local estimate of the demand. This issue has been previously analyzed by Baumol and Quandt, 1964, Puu 1995, Naimzada and Ricchiuti, 2008, Askar, 2013, Askar, 2014. Bounded rational players (firms) update their production strategies based on discrete time periods and by using a local estimate of the marginal profit. With such local adjustment mechanism, the players are not requested to have a complete knowledge of the demand and the cost functions (Agiza, Elsadany, 2004, Naimzada, Sbragia, 2006, Zhang *et al,* 2007, Askar, 2014). The paper is organized as follows: In Section 2, the dynamics of the duopoly game with heterogeneous expectations, linear demand and a quadratic cost function including emission costs is analyzed. The existence and local stability of the equilibrium points are also analyzed. In Section 3 numerical simulations are used to show complex dynamics via computing Lyapunov numbers, and sensitive dependence on initial conditions.

## 2. THE GAME

### 2.1 The construction of the game

The two firms offer their products at discrete-time periods (t = 0,1,2,…) on a common market. We consider a simple Cournot-type duopoly market where firms (players) produce differentiated goods and their production decisions are taken at discrete-time periods (t = 0,1,2,…).

In this study we consider heterogeneous players and more specifically, we consider that the Firm 1 chooses the production quantity in a rational way, following an adjustment mechanism (bounded rational player), while the Firm 2 decides with naïve way by selecting a quantity that maximizes its output (naïve player). At each period t, every firm must form an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period t+1. We suppose that $q_1$ , $q_2$ are the production quantities of each firm, then the inverse demand function (as a function of quantities) is given by the following equation:

$$p_i = a - q_i - dq_j, i \neq j$$

where $p_i$ is the product price of firm i.

So, we have for each firm the following equations:

$$p_1 = a - q_1 - dq_2 \quad \text{and} \quad p_2 = a - q_2 - dq_1 \tag{1}$$

where α is a positive parameter and $d \in (-1,1)$ is the parameter that reveals the differentiation degree of products. For example, if $d = 0$ then both products are independently and each firm participates in a monopoly. But, if $d = 1$ then one product is a substitute for the other, since the products are homogeneous. It is understood that for positive values of the parameter d the larger the value, the less diversification we have in both products. On the other hand negative values of the parameter d are described that the two products are complementary and when $d = -1$ then we have the phenomenon of full competition between the two companies.

We suppose the following cost function:

$$C_i(q_i) = C_p + C_e \tag{2}$$

where

$$C_{pi}(q_i) = cq_i^2 \tag{3}$$

is the quadratic production cost function for firm i, and c > 0 is the same marginal cost for two firms.

Also,

$$C_{ei} = p_c \cdot \varepsilon \cdot q_i \tag{4}$$

is the linear fumes emission cost function for each player, where $p_c$ is the emission license price which is decided from the Government, $\varepsilon \in [0,1]$ is a positive coefficient which is common for two firms and when it is multiplied with each production quantity, it gives the total emissions.

With these assumptions, the profits of the firms are given by:

$$\mathrm{P}_1(q_1, q_2) = p_1 q_1 - C_1(q_1) = (\alpha - q_1 - dq_2) q_1 - cq_1^2 - p_c \cdot \varepsilon \cdot q_1 \tag{5}$$

and

$$\mathrm{P}_2(q_1, q_2) = p_2 q_2 - C_2(q_2) = (\alpha - q_2 - dq_1) q_2 - cq_2^2 - p_c \cdot \varepsilon \cdot q_2 \tag{6}$$

Then the marginal profits at the point of the strategy space are given by:

$$\frac{\partial \mathrm{P}_1}{\partial q_1} = a - p_c \cdot \varepsilon - 2(1+c) q_1 - dq_2 \quad \text{and} \quad \frac{\partial \mathrm{P}_2}{\partial q_2} = a - p_c \cdot \varepsilon - 2(1+c) q_2 - dq_1 \tag{7}$$

To make the calculations easily we set:

$$\mathrm{H} = a - p_c \cdot \varepsilon \tag{8}$$

We suppose that first firm decides to increase its level of adaptation if it has a positive marginal profit, or decreases its level if the marginal profit is negative (bounded rational player). If k > 0 the dynamical equation of the first player is:

$$\frac{q_1(t+1) - q_1(t)}{q_1(t)} = k \frac{\partial \mathrm{P}_1}{\partial q_1} \tag{9}$$

k is the speed of adjustment of player 1, it is a positive parameter which gives the extent of production variation of the firm following a given profit signal. Moreover, it captures the fact that relative effort variations are proportional to the marginal profit.

The second firm decides with naïve way by selecting a production that maximizes its profits (naïve player):

$$q_2(t+1) = \arg\max_y \mathrm{P}_2\left(q_1(t), q_2(t)\right) \tag{10}$$

The dynamical system of the players is described by:

$$\begin{cases} q_1(t+1) = q_1(t) + kq_1(t)\left[\mathrm{H} - 2(1+c)q_1(t) - dq_2(t)\right] \\ q_2(t+1) = \dfrac{\mathrm{H} - dq_1(t)}{2(1+c)} \end{cases} \tag{11}$$

We will focus on the dynamics of this system to the parameter d.

**2.2 Dynamical analysis**

**2.2.1 The equilibriums of the game**

The equilibriums of the dynamical system Eq.(11) are obtained as nonnegative solutions of the algebraic system:

$$\begin{cases} kq_1^*\left[\mathrm{H} - 2(1+c)q_1^* - dq_2^*\right] = 0 \\ q_2^* = \dfrac{\mathrm{H} - dq_1^*}{2(1+c)} \end{cases} \tag{12}$$

which obtained by setting $q_1(t+1) = q_1(t) = q_1^*$ and $q_2(t+1) = q_2(t) = q_2^*$ .

- If $q_1^* = 0$ , then $q_2^* = \dfrac{\mathrm{H}}{2(1+c)}$ and we have the boundary equilibrium:

$$E_0 = \left(0, \frac{\mathrm{H}}{2(1+c)}\right) \tag{13}$$

- If $\mathrm{H} - 2(1+c)q_1^* - dq_2^* = 0$, then we form the following system:

$$\begin{cases} q_1^* = \dfrac{\mathrm{H} - dq_2^*}{2(1+c)} \\ q_2^* = \dfrac{\mathrm{H} - dq_1^*}{2(1+c)} \end{cases} \tag{14}$$

The system's solutions are:

$$q_1^* = q_2^* = \frac{\mathrm{H}}{2(1+c)+d} \ ,$$

and the Nash equilibrium:

$$E_* = \left(q_1^*, q_2^*\right) = \left(\frac{\mathrm{H}}{2(1+c)+d}, \frac{\mathrm{H}}{2(1+c)+d},\right) \tag{15}$$

**2.2.2 Stability of equilibriums**

The study of the local stability of the equilibrium is based on the localization on the complex plane of the eigenvalues of the Jacobian matrix of the dimensional map (Eq.(11)). In order to study the local stability of equilibrium points of the model (11), we consider the Jacobian matrix $J\left(q_1, q_2\right)$ along the variable strategy $\left(q_1, q_2\right)$:

$$J\left(q_1, q_2\right) = \begin{bmatrix} f_{q_1} & f_{q_2} \\ g_{q_1} & g_{q_2} \end{bmatrix} \tag{16}$$

where

$$\begin{aligned} f\left(q_1, q_2\right) &= q_1 + k \cdot q_1 \frac{\partial \mathrm{P}_1}{\partial q_1} = q_1 + k \cdot q_1 \left[\mathrm{H} - 2(1+c) q_1 - d q_2\right] \\ g\left(q_1, q_2\right) &= \frac{\mathrm{H} - d q_1}{2(1+c)} \end{aligned} \tag{17}$$

and we find the Jacobian matrix:

$$J\left(q_1, q_2\right) = \begin{bmatrix} 1 + k\left[\mathrm{H} - 4(1+c) q_1 - d q_2\right] & -dkq_1 \\ -\dfrac{d}{2(1+c)} & 0 \end{bmatrix} \tag{18}$$

For equilibrium $E_0$ we have:

$$J\left(E_0\right) = \begin{bmatrix} 1 + k\mathrm{H}\left(1 - \dfrac{d}{2(1+c)}\right) & 0 \\ -\dfrac{d}{2(1+c)} & 0 \end{bmatrix} \tag{19}$$

with

$$Tr\left[J\left(E_0\right)\right] = 1 + k\mathrm{H}\left[1 - \frac{d}{2(1+c)}\right] \text{ and } Det\left[J\left(E_0\right)\right] = 0 \ .$$

The characteristic equation of $J\left(E_0\right)$ is:

$$l^2 - Tr \cdot l + Det = 0 \tag{20}$$

and the eigenvalues of the Jacobian matrix are:

$$l_1 = 0 \qquad \text{and} \qquad l_2 = 1 + k\mathrm{H}\left[1 - \frac{d}{2(1+c)}\right] \tag{21}$$

Since $\frac{d}{2(1+c)} < 1$ , it's clearly that $\left|l_2\right| > 1$, and the point $E_0$ is unstable.

In the Nash equilibrium point $E_*$ the Jacobian matrix is:

$$J\left(E_*\right) = \begin{bmatrix} 1 - 2k\left(1+c\right)q_1^* & -kdq_1^* \\ -\dfrac{d}{2(1+c)} & 0 \end{bmatrix} \tag{22}$$

The equilibrium point is locally asymptotically stable if

$$\begin{aligned} &i) \quad 1 - Det > 0 \\ &ii) \quad 1 - Tr + Det > 0 \\ &iii) \quad 1 + Tr + Det > 0 \end{aligned} \tag{23}$$

Since

$$1 - Det = 1 + \frac{d^2 k\mathrm{H}}{2(1+c)} q_1^* > 0 \ , \tag{24}$$

and

$$1 - Tr + Det = \frac{kq_1^*}{2(1+c)}\left[2\left(1+c\right) + d\right] \cdot \left[2\left(1+c\right) - d\right] > 0 \tag{25}$$

The conditions (i) and (ii) of Eq.(23) are always satisfied and then the condition (iii) is the condition for the local stability of the Nash Equilibrium. This condition becomes:

$$\begin{aligned} &1 + Tr + Det > 0 \iff \\ &-d^2 k\mathrm{H} + 4\left(1+c\right)d + 4\left(1+c\right)^2 \cdot \left(2 - kH\right) > 0 \end{aligned} \tag{26}$$

The discriminant of Eq.(26) is positive if and only if:

$$\begin{aligned} &\Delta_1 = 16\left(1+c\right)^2 \cdot \left[-\left(k\mathrm{H}\right)^2 + 2k\mathrm{H} + 1\right] > 0 \iff \\ &-\left(k\mathrm{H}\right)^2 + 2k\mathrm{H} + 1 > 0 \iff \\ &k\mathrm{H} \in \left(0, 1 + \sqrt{2}\right) \end{aligned} \tag{27}$$

It follows that Eq.(26) is verified if and only if:

$$\begin{cases} k\mathrm{H} \in \left(0, 1+\sqrt{2}\right) \\ d \in \left(d_1, d_2\right) \end{cases} \tag{28}$$

where

$$d_{1,2} = \frac{-4(1+c) \pm 4(1+c)\sqrt{-(k\mathrm{H})^2 + 2k\mathrm{H} + 1}}{-2k\mathrm{H}} \tag{29}$$

the real roots of Eq. (26). It follows that:

**Proposition:** *The Nash equilibrium of the dynamical system Eq. (11) is locally stable if and only if:*

$$\begin{cases} k\mathrm{H} \in \left(0, 1+\sqrt{2}\right) \\ d \in \left(d_1, d_2\right) \end{cases}$$

### 2.2.3 Numerical simulations

To provide some numerical evidence for the chaotic behavior of the system Eq.(11), as a consequence of change in the parameter d of the product differentiation degree. We present various numerical results to show the chaoticity, including its bifurcations diagrams, strange attractor, Lyapunov numbers and sensitive dependence on initial conditions (Kulenovic, M., Merino, O., 2002). In order to study the local stability properties of the equilibrium points, it is convenient to take the parameters values as follows:

$$a = 10,\ k = 0.27, c = 2,\ p_c = 4,\ \varepsilon = 0.5,\ \left(\mathrm{H} = 8\right).$$

Then, $k\mathrm{H} = 2{,}16 \in \left(0\ ,\ 1+\sqrt{2}\right)$ and $d_1 \approx 0.5,\ d_2 \approx 5.02 > 1$ and the stability condition becomes:

$$d \in \left(d_1, d_2\right) \cap \left(-1, 1\right)\ \Rightarrow\ \mathrm{d} \in \left(d_1, 1\right) \tag{30}$$

Numerical experiments are computed to show the bifurcation diagram with respect to d, strange attractors of the system Eq.(11) in the phase plane $\left(q_1, q_2\right)$ and Lyapunov numbers.

Figure 1 shows the bifurcation diagrams with respect to the parameter d. In this figure, one observes complex dynamic behavior such as cycles of higher order and chaos. Figure 3 shows the Lyapunov numbers of the orbit of (0.01,0.01) for $a = 10,\ k = 0.27, c = 2,\ p_c = 4,\ \varepsilon = 0.5,\ \left(\mathrm{H} = 8\right)$, $d = -0.9$ (left) and $d = -0.99$ (right). If the Lyapunov number is greater of 1, one has evidence for chaos. Figure 3 shows the graphs of the same orbit (strange attractors) for $a = 10,\ k = 0.27, c = 2,\ p_c = 4,\ \varepsilon = 0.5,\ \left(\mathrm{H} = 8\right)$, $d = -0.9$ (left) and $d = -0.99$ (right). From these results when all parameters are fixed and only d is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors. To demonstrate the sensitivity to initial conditions of the system Eq. (11) we compute two orbits with initial points (0.01,0.01) and (0.0101,0.0101), respectively. Figure 5 shows sensitive dependence on initial conditions for x-coordinate of the two orbits, for the system Eq.(11), plotted against the time with the parameter values $a = 10,\ c = 2,\ k = 0.27,\ p_c = 0.75,\ \varepsilon = 0.5,\ (\mathrm{H} = 8),\ d = -0.9$. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly. From Figure 4 we show that the time series of the system Eq.(11) is sensitive dependence to initial conditions, i.e. complex dynamics behavior occur in this model.

If $a = 10,\ c = 2,\ k = 0.2,\ p_c = 4,\ \varepsilon = 0.5,\ (\mathrm{H} = 8)$ (we changed the value of parameter k), $k\mathrm{H} = 1.6 \in \left(0\ ,\ 1 + \sqrt{2}\right)$ and $d_1 \approx -1.052... < -1,\ d_2 \approx 8.552... > 1$. From Eq. (28) it follows that for each d in the internal (-1,1) the Nash equilibrium is locally asymptotically stable (Figure 5). Therefore, in the case of quadratic costs there are stable trajectories and a higher (lower) degree of product differentiation does not to destabilize (stabilize) the market.

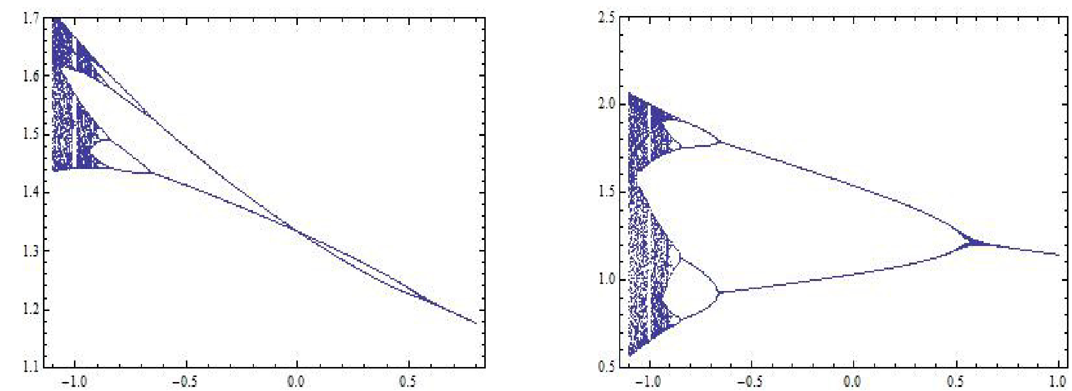

**Figure 1:** Bifurcation diagrams with respect to the parameter d against variable $q_1$ (left) and $q_2$ (right), with 400 iterations of the map Eq. (11) for $a = 10,\ c = 2,\ k = 0.27,\ p_c = 4,\ \varepsilon = 0.5,\ (\mathrm{H} = 8)$.

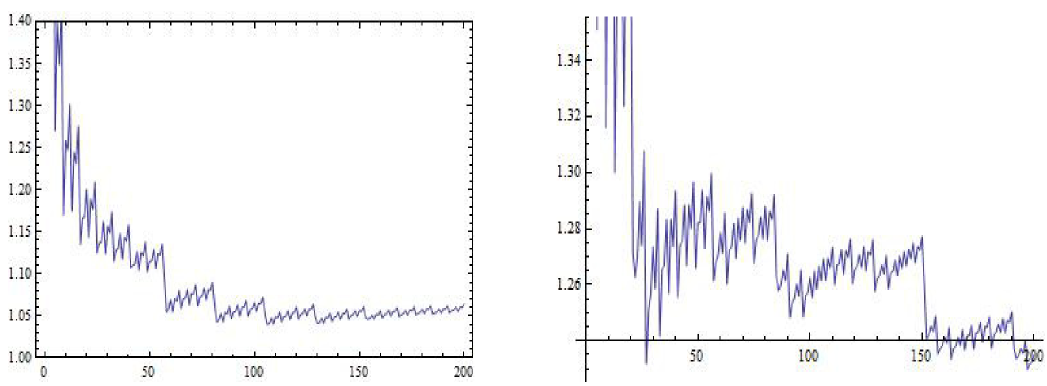

**Figure 2:** Lyapunov numbers of the orbit of the point A(0.1,0.1) versus the number of iterations for $a = 10,\ c = 2,\ k = 0.27,\ p_c = 4,\ \varepsilon = 0.5,\ (\mathrm{H} = 8)$, and for $d = -0.9$ (left) and $d = -0.99$ (right).

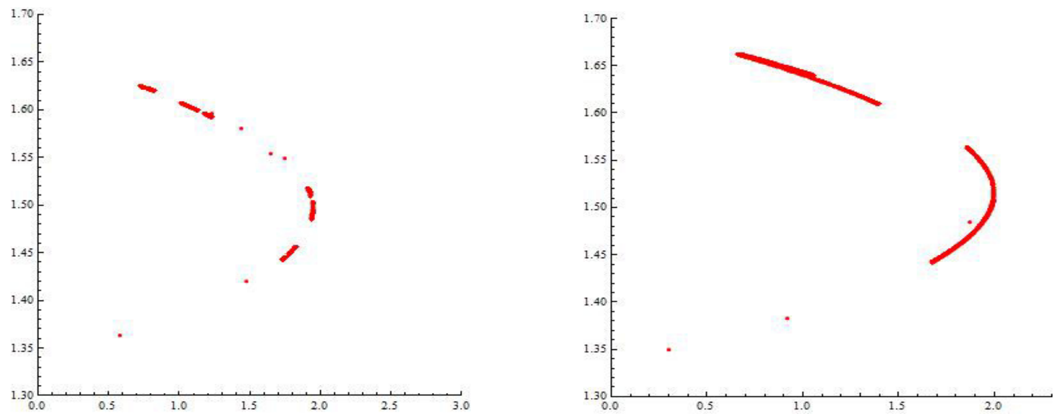


**Figure 3:** Phase portrait (strange attractors). The orbit of (0.1,0.1) with 2000 iterations of the map Eq.(11) for $a = 10,\ c = 2,\ k = 0.27,\ p_c = 4,\ \varepsilon = 0.5,\ (\mathrm{H} = 8)$, and for $d = -0.9$ (left) and $d = -0.99$ (right).

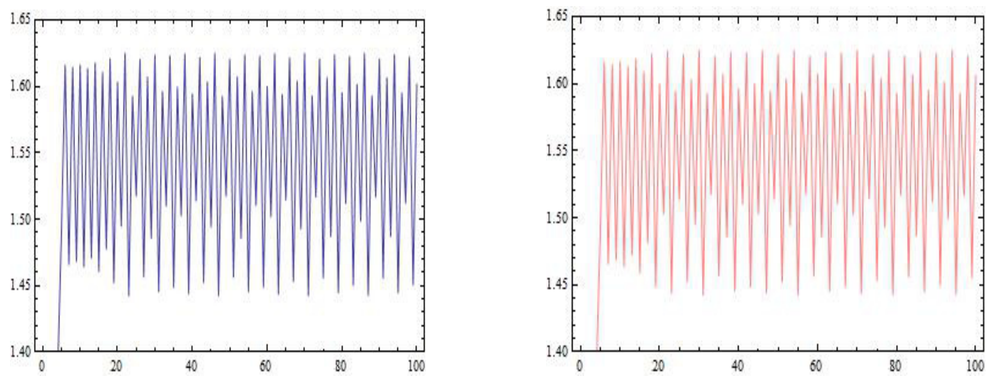


**Figure 4:** Sensitive dependence on initial conditions for x-coordinate plotted against the time: the two orbits: the orbit of (0.01,0.01) (left) and the orbit of (0.0101,0.0101) (right), for the system Eq.(11), with the parameters values $a = 10,\ c = 2,\ k = 0.27,\ p_c = 4,\ \varepsilon = 0.5,\ (\mathrm{H} = 8),\ d = -0.9$.

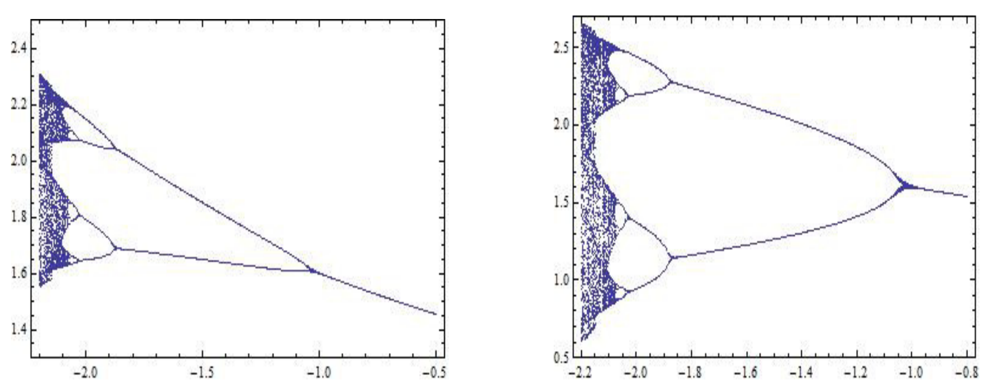


**Figure 5:** Bifurcation diagrams with respect to the parameter d against variable $q_1$ (left) and $q_2$ (right), with 400 iterations of the map Eq. (11) for $a = 10,\ c = 2,\ k = 0.2,\ p_c = 4,\ \varepsilon = 0.5,\ (\mathrm{H} = 8)$.

## 3. CONCLUSION

In this paper we analyzed the dynamics of a nonlinear Cournot- type duopoly game with differentiated goods, linear demand and a cost function that includes emission costs. Existence and stability of equilibria of this system are studied. We proved that the parameter of horizontal product differentiation may change the stability of equilibrium and cause a structure to behave chaotically. For some values of this parameter there is a stable Nash equilibrium. Decreasing these values, the equilibrium becomes unstable, through period-doubling bifurcation.

The chaotic features are justified numerically via computing Lyapunov numbers and sensitive dependence on initial conditions. Also, we show that in our case of quadratic costs there are stable trajectories for each d in the internal (-1,1) and a higher (lower) degree of product differentiation does not tend to destabilize the market.

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